

The University of Chicago

METRIC GEOMETRY OF SURFACES
IN FOUR-DIMENSIONAL SPACE

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1938

By

CHARLES EUGENE SPRINGER

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METRIC GEOMETRY OF SURFACES IN FOUR-DIMENSIONAL SPACE

I. INTRODUCTION

In this thesis a study is made of the metric differential geometry of an ordinary surface V_2 immersed in a linear space of four dimensions S_4 . The point of departure is the system of defining differential equations for the surface as developed by Eisenhart.¹ Sections IV and V contain, among other things, some extensions to the general surface in S_4 of certain results found by Grove² for a surface in space S_4 which sustains an orthogonal conjugate net. Moreover, some of the results in the first five sections have been obtained independently by Heyda, a student of Grove, in his Master's thesis.

Some of the principal results of the present investigation are disclosed by the following remarks. In Section V the locus of the centers of principal curvature of all normal projection surfaces, which are defined in the work of Grove, is found to be a conic, which is called the central conic for the surface V_2 at the point P_x . Heyda obtains this locus, but a briefer derivation of it is afforded here by the use of the tensor notation. Furthermore, certain geometric properties of the surface V_2 are revealed by a study of the central conic. The hyperplane,

¹L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1926, p. 189.

²V. G. Grove, Trans. Amer. Math. Soc., Vol. 39 (1936), p. 60. This paper is referred to hereafter simply as Grove.

which contains the osculating plane to a curve on the surface at the point P_x and the tangent plane to the surface at P_x , is considered in Section VI. There are, in general, two curves on the surface through a point P_x for which these hyperplanes intersect the normal plane to the surface in the same line. This gives rise to the limiting intersector normals, which, in turn, lead to the limiting intersector net which is a covariant net on the surface. In Section VII two covariant reciprocal cubic cones, at a point on the surface, are derived, and the condition that these cones coincide at every point of the surface is later interpreted geometrically. Section VIII exhibits the result that a certain congruence is conjugate to the surface V_2 if, and only if, the sum of the squares of the mean curvatures of the normal projection surfaces, determined by orthogonal normals to the surface, is a constant. The osculating hyperplane to a curve on the surface is introduced in Section IX. This leads to a study of the curves on the surface which are quasi-asymptotic geodesics, and also the curves which are geodesics for which the osculating hyperplane at the point P_x contains the normal plane to the surface at P_x .

The tensor notation is convenient when the discussion concerns the general surface, but the classic notation is more feasible in certain instances. Hence, a change from one notation to the other will be effected when it appears desirable. Certain conventions as to notation will be adhered to throughout the work. Latin superscripts and subscripts will be supposed to have the range 1, 2. The Greek letter α will always take the range 1, 2, 3, 4. Greek letters near the end of the alphabet will have the range 3, 4. A repeated index will indicate summation over the assigned range as given above, unless special mention is made of

the fact that the repeated index does not indicate summation. The table of formulas, which will enable us to pass from the tensor notation to the more classic notation, is given here for future reference:

$$\begin{aligned}
 &g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G; \\
 (A) \quad &b(\sigma)_{11} = D_\sigma, \quad b(\sigma)_{12} = b(\sigma)_{21} = D_\sigma', \quad b(\sigma)_{22} = D_\sigma'' \quad (\sigma = 3, 4); \\
 &\nu_{(43)1} = A_3, \quad \nu_{(43)2} = B_3, \quad \nu_{(34)1} = A_4, \quad \nu_{(34)2} = B_4; \\
 &u^1 = u, \quad u^2 = v.
 \end{aligned}$$

The classic notation will hereafter be designated as "notation (A)".

We shall use $g_{12} = g_{21} = F = 0$ in the entire paper. If also $D_3' = D_4' = 0$, the surface V_2 sustains an orthogonal conjugate net which is the parametric net.¹

¹Grove, loc. cit., p. 60.

II. THE DEFINING DIFFERENTIAL EQUATIONS AND THEIR INTEGRABILITY CONDITIONS

Let us consider a surface V_2 represented by the equations

$$(1) \quad x^\alpha = x^\alpha(u^1, u^2) \quad (\alpha = 1, \dots, 4)$$

in space S_4 , the functions here involved being assumed to be analytic and not capable of being expressed as functions of a single variable. Let the linear element on the surface be given by

$$(2) \quad ds^2 = g_{ij} du^i du^j,$$

where

$$g_{ij} = \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\alpha}{\partial u^j}.$$

We shall call the plane containing all the normal lines to the surface V_2 at the point P_X on the surface the normal plane to V_2 at P_X . Let us select, in the normal plane to V_2 at P_X , two unit normal vectors with components $\eta_{(\sigma)}^\alpha$. Since x^α is invariant under a change of coordinates on the surface V_2 , the ordinary derivative $\partial x^\alpha / \partial u^i$ can be replaced by the covariant derivative denoted by $x^\alpha_{;i}$. Then we have

$$(3) \quad \eta_{(\sigma)}^\alpha x^\alpha_{;i} = 0, \quad \eta_{(\sigma)}^\alpha \eta_{(\tau)}^\alpha = \delta_\sigma^\tau,$$

where δ_σ^τ is the ordinary Kronecker delta.

The functions $x^\alpha(u^1, u^2)$ and $\eta_{(\sigma)}^\alpha(u^1, u^2)$ satisfy the following system of differential equations

$$(4) \quad \begin{aligned} x^\alpha_{,ij} &= b_{(\sigma)ij} \eta^\alpha_{(\sigma)} , \\ \eta^\alpha_{(\sigma),j} &= -b_{(\sigma)ij} g^{im} x^\alpha_{,m} + \nu'_{(\tau\sigma)j} \eta^\alpha_{(\tau)} , \end{aligned}$$

where the left member of the first of these equations denotes the second covariant derivative with respect to the fundamental quadratic form (2) and where, for each σ and τ , the quantities $\nu'_{(\tau\sigma)j}$ are the covariant components of a vector subject to the conditions¹

$$(5) \quad \nu'_{(\tau\sigma)j} + \nu'_{(\sigma\tau)j} = 0 , \quad \nu'_{(\sigma\sigma)j} = 0 .$$

It is to be recognized here that $\nu'_{(\tau\sigma)j} = \eta^\alpha_{(\tau)} \eta^\alpha_{(\sigma),j}$ and that $\eta^\alpha_{(\sigma),j}$ is the ordinary derivative $\partial \eta^\alpha_{(\sigma)} / \partial u^j$.

The integrability conditions for the system (4), that is, the Gauss and Codazzi equations for the surface V_2 , are¹

$$(6) \quad R_{1212} = b_{(\sigma)11} b_{(\sigma)22} - b_{(\sigma)12} b_{(\sigma)21} ,$$

$$(7) \quad b_{(\sigma)1j,k} - b_{(\sigma)1k,j} = \nu'_{(\tau\sigma)\kappa} b_{(\tau)ij} - \nu'_{(\tau\sigma)j} b_{(\tau)i\kappa} ,$$

$$(8) \quad \nu'_{(\tau\sigma)j,\kappa} - \nu'_{(\tau\sigma)\kappa,j} + g^{im} (b_{(\tau)ij} b_{(\sigma)m\kappa} - b_{(\tau)i\kappa} b_{(\sigma)mj}) = 0 .$$

For convenience of reference we shall rewrite these integrability conditions using notation (A). With the supposition that the parametric curves are orthogonal, equations (6), (7), and (8) assume the form

$$(6') \quad G_u \left(\frac{E_u}{E} + \frac{G_u}{G} \right) + E_v \left(\frac{E_v}{E} + \frac{G_v}{G} \right) = 2(E_{vv} + G_{uu}) + 4(D_3 D_3'' + D_4 D_4'' - D_3'^2 - D_4'^2) ;$$

¹Eisenhart, loc. cit. p. 189.

$$\begin{aligned}
 D_{3v} - D'_{3u} + B_4 D_4 - A_4 D'_4 &= \frac{E_v}{2} \left(\frac{D_3}{E} + \frac{D_3''}{G} \right) - \frac{D_3'}{2} \left(\log \frac{E}{G} \right)_u, \\
 D_{4v} - D'_{4u} + B_3 D_3 - A_3 D'_3 &= \frac{E_v}{2} \left(\frac{D_4}{E} + \frac{D_4''}{G} \right) - \frac{D_4'}{2} \left(\log \frac{E}{G} \right)_u, \\
 (7') \quad D''_{3u} - D'_{3v} + A_4 D_4'' - B_4 D'_4 &= \frac{G_u}{2} \left(\frac{D_3}{E} + \frac{D_3''}{G} \right) + \frac{D_3'}{2} \left(\log \frac{E}{G} \right)_v, \\
 D''_{4u} - D'_{4v} + A_3 D_3'' - B_3 D'_3 &= \frac{G_u}{2} \left(\frac{D_4}{E} + \frac{D_4''}{G} \right) + \frac{D_4'}{2} \left(\log \frac{E}{G} \right)_v;
 \end{aligned}$$

$$(8') \quad A_{3v} - B_{3u} + \frac{1}{E}(D_4 D_3' - D_4' D_3) + \frac{1}{G}(D_4' D_3'' - D_4'' D_3') = 0.$$

If $D_3' = D_4' = 0$, these equations reduce to the integrability conditions given by Grove.¹

¹Grove, loc. cit., p. 61.

III. POWER SERIES EXPANSIONS FOR A LOCAL COORDINATE SYSTEM

It is convenient to use an orthogonal system of parametric curves on the surface so that $x^\alpha_{,1} x^\alpha_{,j} = 0$ ($1 \neq j$). Let us introduce unit vectors $\eta^\alpha_{(i)}$ along the tangents to the parametric curves, so that

$$(9) \quad \eta^\alpha_{(1)} \equiv \frac{x^\alpha_{,1}}{\sqrt{g_{11}}} \quad , \quad \eta^\alpha_{(2)} \equiv \frac{x^\alpha_{,2}}{\sqrt{g_{22}}} \quad .$$

We shall use the directions with components $\eta^\alpha_{(i)}$ ($i = 1, 2$) and $\eta^\alpha_{(\sigma)}$ ($\sigma = 3, 4$) as local axes of reference, and ξ^α as the local coordinates of the point $P(x + \Delta x)$. Then, if y^α are the cartesian coordinates of the point $P(x + \Delta x)$ in space S_4 , we have

$$(10) \quad y^\alpha = x^\alpha + \xi^i \eta^\alpha_{(i)} + \xi^\sigma \eta^\alpha_{(\sigma)} \quad .$$

Again, if y^α are the coordinates of the point on V_2 with curvilinear coordinates $(u^1 + \Delta u^1, u^2 + \Delta u^2)$, where (u^1, u^2) are the curvilinear coordinates of the point P_x , one has

$$(11) \quad y^\alpha = x^\alpha + \frac{\partial x^\alpha}{\partial u^i} \Delta u^i + \frac{1}{2} \frac{\partial^2 x^\alpha}{\partial u^i \partial u^j} \Delta u^i \Delta u^j + \frac{1}{6} \frac{\partial^3 x^\alpha}{\partial u^i \partial u^j \partial u^k} \Delta u^i \Delta u^j \Delta u^k + \dots$$

From (4), on replacing the covariant derivative $x^\alpha_{,1j}$ by its value, we find

$$(12) \quad \frac{\partial^2 x^\alpha}{\partial u^i \partial u^j} = \Gamma^h_{ij} \frac{\partial x^\alpha}{\partial u^h} + b_{(\sigma)ij} \eta^\alpha_{(\sigma)} = \Gamma^m_{ij} \sqrt{g_{mm}} \eta^\alpha_{(m)} + b_{(\sigma)ij} \eta^\alpha_{(\sigma)} \quad ,$$

where the term $\sqrt{g_{mm}}$ is not summed, but the subscript m takes the same value in this term that it takes in the sum $\Gamma^m_{ij} \eta^\alpha_{(m)}$. We

also calculate

$$(13) \quad \frac{\partial^3 x^\alpha}{\partial u^i \partial u^j \partial u^k} = J_{ijk}^m x_{,m}^\alpha + I_{ijk}^\sigma \eta_{(\sigma)}^\alpha$$

and

$$(14) \quad \frac{\partial^4 x^\alpha}{\partial u^i \partial u^j \partial u^k \partial u^l} = P_{ijkl}^m x_{,m}^\alpha + Q_{ijkl}^\sigma \eta_{(\sigma)}^\alpha, \quad ,$$

where

$$(15) \quad \begin{aligned} J_{ijk}^m &\equiv \Gamma_{ij}^h \Gamma_{hk}^m + \frac{\partial \Gamma_{ij}^m}{\partial u^k} - b_{(\sigma)ij} b_{(\sigma)lk} g^{lm}, \\ I_{ijk}^\sigma &\equiv \Gamma_{ij}^h b_{(\sigma)hk} + \frac{\partial b_{(\sigma)ij}}{\partial u^k} + b_{(\tau)ij} \nu_{(\sigma\tau)k}, \end{aligned}$$

and

$$(16) \quad \begin{aligned} P_{ijkl}^m &\equiv \frac{\partial J_{ijk}^m}{\partial u^l} + J_{ijk}^h \Gamma_{hl}^m - I_{ijk}^\sigma b_{(\sigma)pl} g^{pm}, \\ Q_{ijkl}^\sigma &\equiv \frac{\partial I_{ijk}^\sigma}{\partial u^l} + I_{ijk}^\tau \nu_{(\sigma\tau)l} + J_{ijk}^m b_{(\sigma)m l}. \end{aligned}$$

On using the expressions (12), (13), and (14) in (11), and making use of (10), we find for the local coordinates of the point $P(x + \Delta x)$ the series

$$(17) \quad \begin{aligned} \xi^m &= \sqrt{g_{mm}} \left(\Delta u^m + \frac{1}{2} \Gamma_{ij}^m \Delta u^i \Delta u^j + \frac{1}{6} J_{ijk}^m \Delta u^i \Delta u^j \Delta u^k + \frac{1}{24} P_{ijkl}^m \Delta u^i \Delta u^j \Delta u^k \Delta u^l + \dots \right), \\ \xi^\sigma &= \frac{1}{2} b_{(\sigma)ij} \Delta u^i \Delta u^j + \frac{1}{6} I_{ijk}^\sigma \Delta u^i \Delta u^j \Delta u^k + \frac{1}{24} Q_{ijkl}^\sigma \Delta u^i \Delta u^j \Delta u^k \Delta u^l + \dots \end{aligned}$$

From the first of (17) we may write

$$\Delta u^m = \frac{\xi^m}{\sqrt{g_{mm}}}$$

and substitute into the second series of (17) to find

$$(18) \quad \xi^\sigma = \frac{1}{2} \frac{b_{(\sigma)ij}}{\sqrt{g_{ii}g_{jj}}} \xi^i \xi^j + \frac{1}{6} \frac{I_{ijk}^\sigma}{\sqrt{g_{ii}g_{jj}g_{kk}}} \xi^i \xi^j \xi^k + \frac{1}{24} \frac{Q_{ijkl}^\sigma}{\sqrt{g_{ii}g_{jj}g_{kk}g_{ll}}} \xi^i \xi^j \xi^k \xi^l + \dots$$

For $\sigma = 3$, (18) gives the equation of the orthogonal projection of the surface V_2 onto the hyperplane with the equation $\xi^4 = 0$, and for $\sigma = 4$, (18) represents the orthogonal projection of V_2 onto the hyperplane with the equation $\xi^3 = 0$. These projections are called the normal projection surfaces of V_2 determined by the directions $\eta_{(3)}^\alpha$ and $\eta_{(4)}^\alpha$, respectively.¹ Let us denote these normal projection surfaces by $V_{2(\sigma)}$ ($\sigma = 3, 4$). There is an infinitude of normal projection surfaces determined by the infinitude of normals to V_2 at P_X .

The symbols P_{ijk}^m and Q_{ijkl}^σ will not appear in the sequel but they are given as a matter of record.

¹Grove, loc. cit., p. 60.

IV. THE PRINCIPAL RADII OF NORMAL CURVATURE

Let us denote the principal radii of normal curvature for the surface of normal projection $V_2(\sigma)$ by R_σ and $R_\sigma^!$. Then the total curvature K_σ of $V_2(\sigma)$ at P_X is given by

$$(19) \quad K_\sigma = \frac{1}{R_\sigma R_\sigma^!} = \frac{b_{(\sigma)11}b_{(\sigma)22} - b_{(\sigma)12}b_{(\sigma)21}}{g_{11}g_{22}} \quad (\sigma \text{ not summed})$$

and the mean curvature M_σ of $V_2(\sigma)$ at P_X is given by

$$(20) \quad M_\sigma = \frac{b_{(\sigma)11}}{g_{11}} + \frac{b_{(\sigma)22}}{g_{22}}.$$

In case $D_\sigma^! \equiv b_{(\sigma)12} = 0$, the principal radii of normal curvature for the surface $V_2(\sigma)$ are

$$(21) \quad R_\sigma = \frac{E}{D_\sigma}, \quad R_\sigma^! = \frac{G}{D_\sigma^h}.$$

Referring to the integrability condition (6'), we notice that $K_3 + K_4$ is constant at a point of the surface V_2 , so that the sum of the total curvatures of the normal projection surfaces $V_2(\sigma)$, determined by any two perpendicular normals to V_2 , is a constant at a point of the surface.

Let us choose another set of orthogonal unit vectors in the normal plane to V_2 at P_X defined by

$$(22) \quad \bar{\eta}_{(\tau)}^\alpha = t_\tau^\sigma \eta_{(\sigma)}^\alpha.$$

Since the vectors $\eta_{(\tau)}^\alpha$ are orthogonal to each other, it is seen that

$$\bar{\eta}_{(\tau)}^{\alpha} \bar{\eta}_{(\rho)}^{\alpha} = t_{\tau}^{\sigma} t_{\rho}^{\nu} \eta_{(\sigma)}^{\alpha} \eta_{(\nu)}^{\alpha} = \delta_{\sigma}^{\nu} t_{\tau}^{\sigma} t_{\rho}^{\nu} = t_{\tau}^{\sigma} t_{\rho}^{\sigma} = \delta_{\tau}^{\rho}.$$

Since $b_{(\sigma)ij} = \chi_{,ij}^{\alpha} \eta_{(\sigma)}^{\alpha}$, then

$$(23) \quad \bar{b}_{(\tau)ij} = \chi_{,ij}^{\alpha} \bar{\eta}_{(\tau)}^{\alpha} = t_{\tau}^{\sigma} b_{(\sigma)ij},$$

and since

$$\nu_{(\tau\sigma)j} = \eta_{(\tau)}^{\alpha} \eta_{(\sigma),j}^{\alpha}$$

we find

$$(24) \quad \bar{\nu}_{(\mu\rho)j} = \bar{\eta}_{(\mu)}^{\alpha} \bar{\eta}_{(\rho),j}^{\alpha} = t_{\mu}^{\tau} \eta_{(\tau)}^{\alpha} (t_{\rho}^{\sigma} \eta_{(\sigma),j}^{\alpha} + \eta_{(\sigma)}^{\alpha} t_{\rho,j}^{\sigma}) = t_{\mu}^{\tau} t_{\rho}^{\sigma} \nu_{(\tau\sigma)j} + t_{\mu}^{\sigma} t_{\rho,j}^{\sigma}.$$

With these new coefficients, the differential equations (4) become

$$(4') \quad \begin{aligned} \chi_{,ij}^{\alpha} &= \bar{b}_{(\tau)ij} \bar{\eta}_{(\tau)}^{\alpha}, \\ \eta_{(\sigma),j}^{\alpha} &= -\bar{b}_{(\sigma)lj} g^{lm} \chi_{,m}^{\alpha} + \bar{\nu}_{(\mu\sigma)j} \bar{\eta}_{(\mu)}^{\alpha}, \end{aligned}$$

with

$$\bar{\nu}_{(\mu\rho)j} + \bar{\nu}_{(\rho\mu)j} = 0.$$

We denote the new normal projection surfaces determined by the directions $\bar{\eta}_{(\tau)}^{\alpha}$ by $\bar{V}_2(\tau)$ and the total curvatures of these surfaces are given by

$$g_{11}g_{22} \bar{K}_{\tau} = \bar{b}_{(\tau)11} \bar{b}_{(\tau)22} - \bar{b}_{(\tau)12} \bar{b}_{(\tau)21},$$

or by

$$(25) \quad g_{11}g_{22}\bar{K}_\tau = t_\tau^\sigma t_\tau^\rho \begin{vmatrix} b_{(\sigma)11} & b_{(\rho)12} \\ b_{(\sigma)21} & b_{(\rho)22} \end{vmatrix}.$$

If we write (25) in the form

$$g_{11}g_{22}\bar{K}_\tau = c_{\sigma\rho} t_\tau^\sigma t_\tau^\rho \quad (\text{with } \tau = 3 \text{ or } 4)$$

and seek the normals, in the directions of which the normal projection surfaces have maximum and minimum total curvature, we are led to

$$c_{\sigma\rho} t_\tau^\sigma dt_\tau^\rho + c_{\sigma\rho} t_\tau^\rho dt_\tau^\sigma = 0,$$

and thus

$$c_{\sigma\rho} t_\tau^\sigma dt_\tau^\rho + c_{\rho\sigma} t_\tau^\sigma dt_\tau^\rho = 0.$$

Therefore

$$(c_{\sigma\rho} + c_{\rho\sigma}) t_\tau^\sigma dt_\tau^\rho = 0,$$

and, on differentiating the relation $t_\tau^\mu t_\tau^\sigma = \delta_\sigma^\mu$, it is found that

$$t_\tau^\mu dt_\tau^\mu = 0.$$

Elimination of dt_τ^ρ from the last two equations yields

$$\delta_{\rho\mu}^{34} (c_{\sigma\rho} + c_{\rho\sigma}) t_\tau^\sigma t_\tau^\mu = 0.$$

Thus, the total curvature $\bar{K}_\tau$ is maximum and minimum for the normal projection surfaces determined by the directions $\bar{\eta}_{(\rho)}^\alpha = t_\rho^\sigma \eta_{(\sigma)}^\alpha$ if the parameters t_ρ^σ satisfy

$$\delta_{\sigma\tau}^{34} (b_{(\mu)11}b_{(\sigma)22} + b_{(\sigma)11}b_{(\mu)22} - 2b_{(\mu)12}b_{(\sigma)21}) t_\rho^\tau t_\rho^\mu = 0 \quad (\rho = 3 \text{ or } 4).$$

Since the coefficients of t_p^3 t_p^3 and t_p^4 t_p^4 differ only in sign, the two normals which determine normal projection surfaces of maximum and minimum total curvature are perpendicular to each other. These two normals are called the principal normals to V_x at P_x and the surfaces $\bar{V}_x(\tau)$ determined by them are called the principal normal projection surfaces.¹

This question of curvature may be considered as follows. The coordinates of a point P_y on the normal with components $\eta_{(\sigma)}^\alpha$ are given by

$$(26) \quad y^\alpha = x^\alpha + \rho \eta_{(\sigma)}^\alpha.$$

As u^i ($i = 1, 2$) vary over the surface V_x , the point P_y varies, and if P_y moves tangentially to the normal $\eta_{(\sigma)}^\alpha$ it is necessary that

$$dy^\alpha = \kappa \eta_{(\sigma)}^\alpha \quad \text{or} \quad dx^\alpha + \rho d\eta_{(\sigma)}^\alpha + \eta_{(\sigma)}^\alpha d\rho = \kappa d\eta_{(\sigma)}^\alpha$$

or

$$(27) \quad (x_{,i}^\alpha + \rho \eta_{(\sigma),i}^\alpha) du^i + \eta_{(\sigma)}^\alpha (d\rho - \kappa) = 0.$$

Now we multiply (27) by $\eta_{(\sigma)}^\alpha$ and sum for α to find

$$(\eta_{(\sigma)}^\alpha x_{,i}^\alpha + \rho \eta_{(\sigma)}^\alpha \eta_{(\sigma),i}^\alpha) du^i + \eta_{(\sigma)}^\alpha \eta_{(\sigma)}^\alpha (d\rho - \kappa) = 0,$$

which yields $\kappa = d\rho$ so that equations (27) become

$$(28) \quad (x_{,i}^\alpha + \rho \eta_{(\sigma),i}^\alpha) du^i = 0.$$

On multiplying (28) by $x_{,j}^\alpha$ there results

$$(29) \quad (x_{,i}^\alpha x_{,j}^\alpha + \rho x_{,j}^\alpha \eta_{(\sigma),i}^\alpha) du^i = 0.$$

But $x_{,i}^\alpha x_{,j}^\alpha$ is g_{ij} and

¹Grove, loc. cit., p. 62.

$$\eta_{(\sigma),i}^{\alpha} = -b_{(\sigma)li} g^{lm} x_{,m}^{\alpha} + \nu_{(\tau)i} \eta_{(\tau)}^{\alpha},$$

so that

$$x_{,j}^{\alpha} \eta_{(\sigma),i}^{\alpha} = -b_{(\sigma)li} g^{lm} g_{jm} = -b_{(\sigma)ij}.$$

Thus, equation (29) becomes

$$(30) \quad (g_{ij} - \rho b_{(\sigma)ij}) du^i = 0,$$

and on eliminating the du^i from equations (30) we have¹

$$(31) \quad |g_{ij} - \rho b_{(\sigma)ij}| = 0.$$

The values of ρ given by (31) are the principal radii of curvature of V_{Σ} at P_X in the direction with components $\eta_{(\sigma)}^{\alpha}$. In the case $g_{12} = g_{21} = 0$, equation (31) gives

$$\left(\frac{1}{\rho_1 \rho_2} \right)_{\sigma} = \frac{|b_{(\sigma)ij}|}{|g_{ij}|}$$

which is the same as (19). Hence, the total curvatures of the normal projection surfaces $V_{\Sigma(\sigma)}$ in the directions determined by $\eta_{(\sigma)}^{\alpha}$ are the same as the total curvature of V_{Σ} in these directions.

¹Eisenhart, loc. cit., p. 193.

V. THE CENTRAL CONIC

We consider next the locus of the centers of principal normal curvature of the normal projection surface $\bar{V}_a(\tau)$ determined by the direction $\bar{\eta}_{(\tau)}^\alpha = t_\tau^\sigma \eta_{(\sigma)}^\alpha$ as t_τ^σ vary. For the direction $\bar{\eta}_{(\tau)}^\alpha$ equation (31) becomes

$$(32) \quad |\bar{g}_{ij} - \rho \bar{b}_{(\tau)ij}| = 0,$$

which may be written in the form

$$(33) \quad |g_{ij} - \rho t_\tau^\sigma b_{(\sigma)ij}| = 0.$$

Now the coordinates of the centers of principal normal curvature in the direction $\bar{\eta}_{(\tau)}^\alpha$ are given by

$$y^\alpha = x^\alpha + \rho \bar{\eta}_{(\tau)}^\alpha = x^\alpha + \rho t_\tau^\sigma \eta_{(\sigma)}^\alpha.$$

The local coordinates of these centers are expressed by

$$(34) \quad \xi^\sigma = \rho t_\tau^\sigma.$$

On eliminating ρt_τ^σ between (33) and (34) we obtain the conic with the equations

$$(35) \quad |g_{ij} - \xi^\sigma b_{(\sigma)ij}| = 0, \quad \xi^1 = 0, \quad \xi^2 = 0.$$

Therefore, the locus of the centers of principal normal curvature for all normal projection surfaces at a point of the surface V_a , is a conic lying in the normal plane to the surface V_a at the point. We propose to call this conic the central conic relative to the point P_x on the surface V_a . It will be seen that some properties of the surface relative to its surrounding space are

portrayed by the geometry of the central conic. In notation (A) the equations (35) become

$$(36) \quad K_3 \xi_3^2 + 2H_{34} \xi_3 \xi_4 + K_4 \xi_4^2 - M_3 \xi_3 - M_4 \xi_4 + 1 = 0, \quad \xi_1 = 0, \quad \xi_2 = 0,$$

where

$$\begin{aligned} EGM_3 &= ED_3'' + GD_3, \\ EGM_4 &= ED_4'' + GD_4, \\ EGK_3 &= D_3 D_3'' - D_3' {}^2, \\ EGK_4 &= D_4 D_4'' - D_4' {}^2, \\ 2EGH_{34} &= D_3 D_4'' + D_4 D_3'' - 2D_3' D_4'. \end{aligned}$$

It is to be understood that ξ^0 is the same as ξ_2 and ξ^σ the same as ξ_σ . There is no distinction here between covariant and contravariant indices since the local axes are mutually orthogonal. It is convenient to use the lower indices here so that " ξ_3 squared" may be written ξ_3^2 . For convenience we shall omit writing $\xi_1 = 0$ and $\xi_2 = 0$ hereafter in Section V when configurations in the normal plane to V_s at P_x are being considered.

It is interesting to study the position of the point P_x on V_s relative to the central conic. The polar line of P_x with respect to the central conic (36) has the equation

$$(38) \quad M_3 \xi_3 + M_4 \xi_4 = 2.$$

On making the equation of the conic (36) homogeneous by means of the equation (38) we find the equation of the tangent lines from P_x to the conic in the form

$$4(K_3 \xi_3^2 + 2H_{34} \xi_3 \xi_4 + K_4 \xi_4^2) - (M_3 \xi_3 + M_4 \xi_4)^2 = 0,$$

which, by use of the expressions (37), may be written as

$$(39) \quad [(ED_3'' - GD_3)\xi_3 + (ED_4'' - GD_4)\xi_4]^2 + 4EG(D_3'\xi_3 + D_4'\xi_4)^2 = 0.$$

This equation shows that the point P_X is inside the central conic at every point of a surface for which the form $E du^2 + G dv^2$ is positive definite. The lines (39) are real only in case the product EG is negative.

If P_X is inside the central conic and not at the center of the conic, there is one, and only one chord of the conic for which P_X is the mid-point. Thus, in this case, there is one, and only one, normal projection surface which is minimal. But if there are two minimal normal projection surfaces at P_X , then every surface of normal projection is minimal, for, in this case, P_X must be at the center of the central conic. It can readily be shown that, if P_X is not at the center of the central conic, the unique normal to V_σ at P_X which determines a minimal normal projection surface, has the equation

$$(40) \quad M_3\xi_3 + M_4\xi_4 = 0,$$

where M_3 and M_4 are the mean curvatures of the normal projection surfaces $V_{\sigma(\sigma)}$ ($\sigma = 3, 4$). The normal (40) is parallel to the polar line of P_X with respect to the central conic.

If the total curvature for a normal projection surface at P_X is zero and the central conic is an hyperbola at this point, then there is one other normal projection surface for which the total curvature is zero, since there are two normals to V_σ at P_X which are parallel to the asymptotes of the central conic. Let us call the normals which are parallel to the asymptotes of the central conic the developable normals. They are real and distinct, real and coincident, or imaginary according as the central conic is an hyperbola, a parabola, or an ellipse. These results may

be verified by a study of equation (25).

It will be shown in Section VI that the parametric curves on the surface V_2 at the point P_X are geodesics if, and only if, $E_V = 0$ and $G_U = 0$, and hence $E_{VV} = 0$ and $G_{UU} = 0$, so that integrability condition (6') reduces to the form $D_3 D_3'' + D_4 D_4'' - D_3'^2 - D_4'^2 = 0$. Therefore, we may state that the conic (36) is an equilateral hyperbola, that is $K_3 + K_4 = 0$, if and only if, the parametric curves at P_X are geodesics. Also we observe that there exist orthogonal normals to V_2 at P_X which determine developable normal projection surfaces if, and only if, the parametric curves on V_2 at P_X are geodesics.

From the first of equations (35) it may be seen that the central conic can be written in the form

$$(41) \quad (D_3 \xi_3 + D_4 \xi_4 - E)(D_3'' \xi_3 + D_4'' \xi_4 - G) - (D_3' \xi_3 + D_4' \xi_4)^2 = 0,$$

which shows that the lines in the normal plane with the equations

$$(42) \quad D_3 \xi_3 + D_4 \xi_4 - E = 0, \quad D_3'' \xi_3 + D_4'' \xi_4 - G = 0$$

are tangent to the conic at their points of intersection with the line

$$(43) \quad D_3' \xi_3 + D_4' \xi_4 = 0.$$

The pole of the line (43) with respect to the central conic is the point in the normal plane with coordinates

$$(44) \quad \xi_3 = \frac{ED_4'' - GD_4}{D_3 D_4'' - D_4 D_3''}, \quad \xi_4 = -\frac{ED_3'' - GD_3}{D_3 D_4'' - D_4 D_3''},$$

and the line joining this point to P_X has the equation

$$(45) \quad (ED_3'' - GD_3) \xi_3 + (ED_4'' - GD_4) \xi_4 = 0.$$

The line (43) intersects the polar line of the point P_x with respect to the central conic in the point with coordinates

$$(46) \quad \xi_3 = \frac{2D_4'}{M_3D_4' - M_4D_3'} , \quad \xi_4 = -\frac{2D_3'}{M_3D_4' - M_4D_3'} .$$

It is readily seen that the lines in the normal plane with equations (38), (43), and (45) form the diagonal triangle of the complete quadrilateral determined by the tangents to the central conic through the two points with coordinates (44) and (46).

We shall study next the conditions under which the central conic is degenerate, and give a geometric characterization of each case which appears. The central conic (36) degenerates into two straight lines in case the determinant

$$\begin{vmatrix} 4(D_3D_3'' - D_3'^2) & 2(D_3D_4'' + D_4D_3'' - 2D_3'D_4') & ED_3'' + GD_3 \\ 2(D_3D_4'' + D_4D_3'' - 2D_3'D_4') & 4(D_4D_4'' - D_4'^2) & ED_4'' + GD_4 \\ ED_3'' + GD_3 & ED_4'' + GD_4 & EG \end{vmatrix} = 0,$$

which reduces to the condition

$$(47) \quad D_3'(ED_4'' - GD_4) - D_4'(ED_3'' - GD_3) = 0.$$

In general, this condition is satisfied only for points on V_2 along a certain curve, namely, the curve for which (47) is the equation. But there are several cases in which the expression (47) vanishes identically. We note first that if (47) holds identically the integrability condition (8') reduces to $A_{3v} = B_{3u}$. Hence, the central conic is degenerate if, and only if, $A_{3v} = B_{3u}$.

The condition (47) is satisfied if

$$(48) \quad ED_\sigma'' - GD_\sigma = 0 \quad (\sigma = 3, 4) .$$

Let us consider equations (18) as the equations of the normal projection surfaces $V_{\Sigma(\sigma)}$ in the form of Monge. On calculating the principal directions in the tangent planes of these surfaces at the point P_x we find the nets

$$(49) \quad EGD'_\sigma du^2 + \sqrt{EG}(ED''_\sigma - GD_\sigma) du dv - EGD'_\sigma dv^2 = 0.$$

Now if (48) holds, the directions given by (49), for $\sigma = 3$ or 4 , separate the parametric directions harmonically. Thus, a sufficient condition that the conic (36) be degenerate is that the principal directions on the normal projection surfaces $V_{\Sigma(\sigma)}$ separate the parametric directions harmonically.

The condition (47) is satisfied at every point of the surface V_{Σ} if D_3' and D_4' are zero at every point of the surface, which (with $F = 0$) means that V_{Σ} sustains an orthogonal conjugate net which is parametric. In this case the central conic is composed of the two lines in the normal plane with the equations $D_3 \xi_3 + D_4 \xi_4 = E$ and $D_3'' \xi_3 + D_4'' \xi_4 = G$.¹ Thus, a sufficient condition that the central conic (36) be degenerate is that the surface V_{Σ} sustain a net which is orthogonal conjugate.

Finally, the condition (47) is fulfilled if the matrix $(D_\sigma, D_\sigma', D_\sigma'')$, ($\sigma = 3, 4$), is of rank one. It will be shown in Section VI that this means that the surface V_{Σ} is immersed in a space of three dimensions.

¹Grove, loc. cit., p. 64.

VI. THE INTERSECTOR NORMAL RELATIVE TO
A GIVEN CURVE ON THE SURFACE

It is necessary at this point to calculate the form of the expansions (17) for the local coordinates of points along the curve C with the equations $u^1 = u^1(s)$ ($i = 1, 2$). Using the notation $u'^1 = du^1/ds$, $u''^1 = d^2u^1/ds^2$, etc., we have, in consequence of (10), (11), (12), and (13),

$$\begin{aligned} y^\alpha &= \chi^\alpha + \xi^m \eta_{(m)}^\alpha + \xi^\sigma \eta_{(\sigma)}^\alpha = \chi^\alpha + \chi_{,i}^\alpha u^i \Delta s + \frac{1}{2} \left[\frac{\partial^2 \chi^\alpha}{\partial u^i \partial u^j} u'^i u'^j + \frac{\partial \chi^\alpha}{\partial u^i} u''^i \right] \Delta s^2 + \\ &\frac{1}{6} \left[\frac{\partial^3 \chi^\alpha}{\partial u^i \partial u^j \partial u^k} u'^i u'^j u'^k + 3 \frac{\partial^2 \chi^\alpha}{\partial u^i \partial u^j} u''^i u'^j + \frac{\partial \chi^\alpha}{\partial u^i} u'''^i \right] \Delta s^3 + \dots \\ &= \chi^\alpha + \chi_{,m}^\alpha u^m \Delta s + \frac{1}{2} \left[(\sqrt{g_{hh}} \Gamma_{ij}^h \eta_{(h)}^\alpha + b_{(\sigma)ij} \eta_{(\sigma)}^\alpha) u'^i u'^j + \chi_{,m}^\alpha u''^m \right] \Delta s^2 + \\ &\frac{1}{6} \left[(\Gamma_{ijk}^\sigma \eta_{(\sigma)}^\alpha + J_{ijk}^m \chi_{,m}^\alpha) u'^i u'^j u'^k + 3(\sqrt{g_{hh}} \Gamma_{ij}^h \eta_{(h)}^\alpha + b_{(\sigma)ij} \eta_{(\sigma)}^\alpha) u''^i u'^j + \chi_{,m}^\alpha u'''^m \right] \Delta s^3 + \dots, \end{aligned}$$

so that the local coordinates (17) may be written as

$$\begin{aligned} \frac{\xi^m}{\sqrt{g_{mm}}} &= u^m \Delta s + \frac{1}{2} (\Gamma_{ij}^m u'^i u'^j + u''^m) \Delta s^2 + \frac{1}{6} (J_{ijk}^m u'^i u'^j u'^k + 3 \Gamma_{ij}^m u''^i u'^j + u'''^m) \Delta s^3 + \dots, \\ (50) \quad \xi^\sigma &= \frac{1}{2} b_{(\sigma)ij} u'^i u'^j \Delta s^2 + \frac{1}{6} (\Gamma_{ijk}^\sigma u'^i u'^j u'^k + 3 b_{(\sigma)ij} u''^i u'^j) \Delta s^3 + \dots \end{aligned}$$

Let us now determine the equations of the osculating plane of the curve C in local coordinates. We suppose the desired osculating plane to have equations of the form

$$(51) \quad \begin{aligned} A_\ell \xi^\ell + A_\tau \xi^\tau &= 0 & (\text{with } \tau = 3), \\ A_\ell \xi^\ell + A_\tau \xi^\tau &= 0 & (\text{with } \tau = 4), \end{aligned}$$

and require that the series (50) satisfy these equations identically to terms of the second order in Δs . That is,

$$(52) \quad A_\ell \sqrt{g_{\ell\ell}} [u^\ell \Delta s + \frac{1}{2} (\Gamma_{ij}^\ell u^i u^j + u^{n\ell}) \Delta s^2] + A_\tau [\frac{1}{2} b_{(\tau)ij} u^i u^j \Delta s^2] \equiv 0,$$

in which τ is not summed but takes the range 3, 4. From (52), it follows that

$$(53) \quad \begin{aligned} A_\ell \xi^\ell + A_\tau \xi^\tau &= 0, \\ A_\ell \sqrt{g_{\ell\ell}} u^\ell &= 0, \\ A_\ell \sqrt{g_{\ell\ell}} (\Gamma_{ij}^\ell u^i u^j + u^{n\ell}) + A_\tau b_{(\tau)ij} u^i u^j &= 0. \end{aligned}$$

Elimination of A_ℓ , A_τ ($\tau = 3$ or 4) from (53) yields the equations of the osculating plane to C in local coordinates, namely

$$(54) \quad \begin{aligned} \xi^4 (u^2 \sqrt{g_{22}} b_{(\tau)ij} u^i u^j) - \xi^2 (u^4 \sqrt{g_{11}} b_{(\tau)ij} u^i u^j) + \\ \sqrt{g_{44} g_{22}} \xi^\tau [u^4 (\Gamma_{ij}^2 u^i u^j + u^{n2}) - u^2 (\Gamma_{ij}^4 u^i u^j + u^{n4})] &= 0. \end{aligned}$$

If we write $\Gamma_{ij}^\ell u^i u^j + u^{n\ell} \equiv c^\ell$, the last bracket expression in (54) may be written as $u^1 c^2 - u^2 c^1$ or $\delta_{\ell m}^{12} u^\ell c^m$, so that the equations of the osculating plane take the form

$$(55) \quad \delta_{\ell m}^{12} [(\xi^\ell u^m \sqrt{g_{mm}}) b_{(\tau)ij} u^i u^j + \sqrt{g_{44} g_{22}} (u^\ell c^m) \xi^\tau] = 0 \quad (\tau = 3, 4).$$

If the equations of the curve C satisfy the differential equations $\Gamma_{ij}^l u^i u^j + u''^l = 0$ ($l = 1, 2$), then C is a geodesic on the surface V_2 . In particular, if the curve Cu^1 ($du^2 = 0$) is a geodesic, then $\Gamma_{11}^2 = 0$, which means that $E_v = 0$ in notation (A). Similarly, if the curve Cu^2 ($du^1 = 0$) is a geodesic, then $\Gamma_{22}^1 = 0$, or $G_u = 0$. Therefore, the parametric curves on the surface V_2 are geodesics if, and only if, $E_v = G_u = 0$. Referring to the integrability condition (6') we note that in this case

$$(56) \quad D_3 D_3'' + D_4 D_4'' - D_3'^2 - D_4'^2 = 0.$$

This result was used in Section V.

Let us find the equation of a hyperplane which contains both the osculating plane to the curve C at P_x and the tangent plane to the surface V_2 at P_x . To accomplish this we multiply equation (55) by $b_{(\sigma)ij} u^i u^j$ ($\sigma \neq \tau$), then rewrite the resulting equation with σ and τ interchanged, and then subtract these equations to obtain

$$(57) \quad (\delta_{lm}^{12} u^l c^m) (\delta_{34}^{\sigma\tau} \xi_\sigma b_{(\tau)ij} u^i u^j) = 0.$$

If the quantities c^m ($m = 1, 2$) are zero, that is, if the curve C is a geodesic on the surface V_2 , then equation (55) reduces to

$$(58) \quad \delta_{lm}^{12} \xi^l u^m \sqrt{g_{mm}} = 0,$$

since the factor $b_{(\tau)ij} u^i u^j$ cannot vanish if the osculating plane to the curve C is determinate. The equation of the hyperplane which contains the osculating plane to the geodesic C may then be written in notation (A) as

$$(59) \quad \xi_2 \sqrt{E} du - \xi_1 \sqrt{G} dv = 0,$$

and this hyperplane obviously contains the normal plane to V_2 at P_x . Thus, the hyperplane (59) contains the normal plane to V_2 at P_x and the osculating plane to C at P_x if, and only if, C is a geodesic, and therefore, the osculating plane to C at P_x intersects the normal plane to V_2 at P_x in a line if, and only if, C is a geodesic. To a given hyperplane of the type (59), i.e., to a given ratio ξ_1/ξ_2 there corresponds a one-parameter family of curves on the surface V_2 which are geodesics. And corresponding to a family of geodesics on the surface V_2 given by $\varphi(u,v) = \text{const.}$, we have the hyperplane

$$\xi_1 \sqrt{G} \varphi_u + \xi_2 \sqrt{E} \varphi_v = 0.$$

It is easy to demonstrate that the cross-ratio of the four tangents to any four geodesics through P_x is equal to the cross-ratio of the four corresponding hyperplanes.

If the curve C is not a geodesic, then the first factor of the left member of equation (57) is not identically zero, so that the hyperplane which contains the osculating plane to C at the point P_x and the tangent plane to the surface V_2 at P_x has the equation

$$(60) \quad \delta_{34}^{\sigma\tau} \xi_\sigma b_{(\tau)} i_j u^i u^j = 0.$$

In notation (A) the line of intersection of the hyperplane (60) and the normal plane to V_2 at P_x has the equations

$$(61) \quad \xi_1 = 0, \xi_2 = 0, (\xi_3 D_4 - \xi_4 D_3) du^2 + 2(\xi_3 D_4' - \xi_4 D_3') du dv + (\xi_3 D_4'' - \xi_4 D_3'') dv^2 = 0.$$

These equations associate two directions on the surface V_2 at P_X with a given normal line to V_2 at P_X . If the ratio ξ_3/ξ_4 is a function of u and v , say $g(u, v)$, a normal line is determined at every point of the surface by the equations $\xi_3/\xi_4 = g(u, v)$, $\xi_1=0$, $\xi_2=0$. Thus, the third of equations (61) represents a net on the surface V_2 corresponding to the double infinity of normals given by $\xi_3/\xi_4 = g(u, v)$, $\xi_1=0$, $\xi_2=0$. A particular instance of this will appear in Section VIII.

We note from equations (61) that the normals corresponding to the parametric directions on the surface V_2 are the lines given by

$$(62) \quad \begin{aligned} \xi_1=0, \quad \xi_2=0, \quad \xi_3 D_4 - \xi_4 D_3 &= 0 ; \\ \xi_1=0, \quad \xi_2=0, \quad \xi_3 D_4'' - \xi_4 D_3'' &= 0 . \end{aligned}$$

Grove¹ calls these lines "the intersector normals of the curves C_u and C_v ", respectively. Let us call the normal with equations (61) "the intersector normal relative to a given curve (with direction dv/du) on the surface V_2 at P_X ."

If the surface V_2 sustains an orthogonal conjugate net, in which case $F = 0$ and $D_3' = D_4' = 0$, then equations (61) reduce to

$$(63) \quad \xi_1=0, \quad \xi_2=0, \quad (\xi_3 D_4 - \xi_4 D_3) du^2 + (\xi_3 D_4'' - \xi_4 D_3'') dv^2 = 0.$$

Therefore, the tangents to the curves of the net on V_2 corresponding to a given system of intersector normal lines, separate the directions of the orthogonal conjugate net harmonically.

¹Grove, loc. cit., p. 65.

If the net (63) is indeterminate, then

$$\xi_3 D_4 - \xi_4 D_3 = 0, \quad \xi_3 D_4'' - \xi_4 D_3'' = 0,$$

so that $D_3 D_4'' - D_4 D_3'' = 0$ and the surface is immersed in space S_3 .

To a given direction dv/du on the surface V_x there corresponds one normal with the equations (61), but for a given normal, that is for a given ratio ξ_3/ξ_4 , there are two corresponding directions given by equations (61). However, these two corresponding directions coincide for the particular normals given by

$$(64) \quad \xi_1 = 0, \xi_2 = 0, \quad (D_4'^2 - D_4 D_4'') \xi_3^2 + (D_3 D_4'' + D_4 D_3'' - 2 D_3' D_4') \xi_3 \xi_4 + (D_3'^2 - D_3 D_3'') \xi_4^2 = 0.$$

We shall refer to these lines in the normal plane as the "limiting intersector normals". For each ratio ξ_3/ξ_4 given by the third equation in (64) we have the unique corresponding direction on the surface V_x given by

$$(65) \quad \frac{dv}{du} = - \frac{\xi_3 D_4' - \xi_4 D_3'}{\xi_3 D_4'' - \xi_4 D_3''}.$$

The limiting intersector normals (64) are orthogonal to the asymptotes of the central conic (36). If the parametric curves on the surface V_x at P_x are geodesics, (56) obtains and in this case the limiting intersector normals are orthogonal, and the central conic is an equilateral hyperbola. Thus, the limiting intersector normals are orthogonal if, and only if, the parametric curves, through the point P_x , are geodesics.

If the limiting intersector normals are coincident we have

$$(66) \quad (D_3 D_4'' + D_4 D_3'' - 2 D_3' D_4')^2 = 4 (D_3'^2 - D_3 D_3'') (D_4'^2 - D_4 D_4''),$$

which can be written in the form

$$(67) \quad (D_3 D_4'' - D_4 D_3'')^2 = 4 (D_3 D_4' - D_4 D_3') (D_3' D_4'' - D_4' D_3'').$$

Thus, the central conic (if not degenerate) is a parabola in case the limiting intersector normals coincide. The unique direction on the surface corresponding to this coincident normal is given by

$$(68) \quad \frac{dv}{du} = \frac{1}{2} \frac{D_4 D_3'' - D_3 D_4''}{D_3' D_4'' - D_4' D_3''},$$

and the corresponding coincident normal has the equations

$$\xi_1 = 0, \xi_2 = 0, (D_3 D_4'' + D_4 D_3'' - 2 D_3' D_4') \xi_3 - 2 (D_3 D_3'' - D_3'^2) \xi_4 = 0,$$

which, by use of (66) and (37), can be written in the form

$$(69) \quad \xi_1 = 0, \xi_2 = 0, \sqrt{K_4} \xi_3 - \sqrt{K_3} \xi_4 = 0.$$

It can readily be verified that this line is parallel to the directrix of the central parabola. Thus, the total curvature of the normal projection surface determined by the normal with the equations

$$(70) \quad \xi_1 = 0, \xi_2 = 0, \sqrt{K_3} \xi_3 + \sqrt{K_4} \xi_4 = 0,$$

is infinite. Therefore, the limiting intersector normals and the developable normals coincide in case the central conic is a parabola.

It can readily be shown that the normal (69) intersects the central parabola in real points only in case

$$M_3 \sqrt{K_3} + M_4 \sqrt{K_4} \geq 2 (K_3 + K_4).$$

In order to find the intersector normal (61) for which the two associated directions at P_X on the surface V_2 are

orthogonal, we calculate the harmonic invariant of the quadratic differential form in equations (61) and the form $E du^2 + G dv^2$ and equate this invariant to zero. Thus,

$$(GD_4 + ED_4'')\xi_3 - (GD_3 + ED_3'')\xi_4 = 0,$$

which, by (37), may be written in the form

$$(71) \quad \xi_1 = 0, \quad \xi_2 = 0, \quad M_4 \xi_3 - M_3 \xi_4 = 0.$$

This normal line is orthogonal to the line (40). Therefore, the intersector normal (61), for which the associated directions on V_2 are orthogonal, is the unique normal orthogonal to the normal which determines a minimal normal projection surface. It is supposed here that P_X is not the center of the central conic.

We shall next obtain a covariant net on V_2 , the directions of which at P_X are given by the correspondence (61) for the limiting intersector normals (64). On eliminating the ratio ξ_3/ξ_4 between (61) and (65), the following net of curves is obtained

$$(72) \quad (D_3 D_4' - D_4 D_3') du^2 + (D_3 D_4'' - D_4 D_3'') du dv + (D_3' D_4'' - D_4' D_3'') dv^2 = 0.$$

Let us call the net (72) the "limiting intersector net on the surface V_2 ". Thus, the curves on the surface whose hyperplanes (60) intersect the normal plane to the surface in the limiting intersector normals, are the curves of the limiting intersector net (72).

This same net may be arrived at in the following way. The third of equations (61) may be written in the form

$$\frac{\xi_4}{\xi_3} = \frac{D_4 + 2 D_4' t + D_4'' t^2}{D_3 + 2 D_3' t + D_3'' t^2},$$

where t is used for the ratio dv/du . It is evident here that if the matrix $(D_\sigma, D_\sigma', D_\sigma'')$ ($\sigma = 3, 4$) is of rank one, then ξ_4/ξ_3 does not depend upon dv/du but is a function of u and v only. Therefore, in this case there is a unique normal to the surface at every point and hence the surface is immersed in a three-dimensional space S_3 . This result was referred to in the last paragraph of Section V.

On differentiating the right member of the last equation with respect to t and equating the result to zero, we find

$$(D_3'D_4'' - D_4'D_3'')t^2 + (D_3D_4'' - D_4D_3'')t + (D_3D_4' - D_4D_3') = 0,$$

and on replacing t by dv/du the net (72) is obtained. Therefore, we have the theorem that the directions on the surface V_2 , for which the ratio ξ_4/ξ_3 , in the correspondence (61), has maximum and minimum values, are given by the net (72).

If in the equation (72) we take $D_3' = D_4' = 0$, this net takes the form

$$(D_3D_4'' - D_4D_3'')du\,dv = 0,$$

so that, if the surface V_2 is not immersed in space S_3 , and if the surface V_2 sustains an orthogonal conjugate net, the limiting intersector net coincides with the orthogonal conjugate net. It may be observed that the net (72) is indeterminate if the surface V_2 is immersed in space S_3 .

To each tangent line to the curves of the net (72) at P_x there corresponds a hyperplane (59) which intersects the tangent plane to V_2 at P_x in a line. The two lines thus associated with the tangents to the curves of the net (72) are obtained by eliminating dv/du between (59) and (72). Thus, the lines in the

tangent plane are found to be given by

$$(73) \quad \xi_3 = 0, \xi_4 = 0, G(D_3 D_4' - D_4 D_3') \xi_1^2 + \sqrt{EG} (D_3 D_4'' - D_4 D_3'') \xi_1 \xi_2 + E (D_3' D_4'' - D_4' D_3'') \xi_2^2 = 0.$$

These lines separate the tangents to the curves of the net (72) harmonically if, and only if,

$$2 (D_3' D_4'' - D_4' D_3'') (D_3 D_4' - D_4 D_3') (E + G) - \sqrt{EG} (D_3 D_4'' - D_4 D_3'')^2 = 0.$$

Now if the central conic at P_x is not degenerate but is a parabola, then (67) holds and the last equation takes the form

$$(D_3 D_4'' - D_4 D_3'')^2 (\sqrt{E} - \sqrt{G})^2 = 0.$$

Therefore, if the surface is not immersed in S_3 and the central conic is a parabola, the lines (73) separate the directions (72) harmonically at a point of the surface if, and only if, $E = G$ at the point.

VII. RECIPROCAL CUBIC CONES AT A POINT ON THE SURFACE

For any direction dv/du on the surface V_2 through the point P_X , the equation (59) represents the hyperplane containing the osculating plane to the geodesic with the direction dv/du at P_X , and the normal plane to the surface at P_X . The third of equations (61) represents the hyperplane which contains the tangent plane to the surface V_2 at P_X and the osculating plane to the geodesic with the direction dv/du . To find the locus of the osculating planes to the geodesics on the surface V_2 at P_X as dv/du varies, we eliminate dv/du between equations (59) and (61) to find a cubic cone with equation

$$(74) \quad G(D_4\xi_3 - D_3\xi_4)\xi_1^2 + 2\sqrt{EG}(D_4'\xi_3 - D_3'\xi_4)\xi_1\xi_2 + E(D_4''\xi_3 - D_3''\xi_4)\xi_2^2 = 0.$$

Now if ξ_1/ξ_2 in (59) is replaced by $-\xi_2/\xi_1$, and ξ_3/ξ_4 in (61) by $-\xi_4/\xi_3$, we obtain hyperplanes which intersect the tangent plane and normal plane, respectively, in lines orthogonal to those considered above. The locus of the plane of intersection of these hyperplanes is the cubic cone with equation

$$(75) \quad E(D_3''\xi_3 + D_4''\xi_4)\xi_1^2 - 2\sqrt{EG}(D_3'\xi_3 + D_4'\xi_4)\xi_1\xi_2 + G(D_3\xi_3 + D_4\xi_4)\xi_2^2 = 0.$$

It can readily be shown that the locus of the normal lines to the tangent hyperplanes of the cone (74) at P_X is the cone (75). This relation between the cones is a reciprocal one, and for this reason the loci (74) and (75) will be referred to as the "reciprocal cubic cones at P_X on the surface V_2 ".

It may be of interest to develop the reciprocal cubic cones in another manner, considering $D_3' = D_4' = 0$ in order to simplify the writing. From the second of equations (53) it is readily observed that $(\sqrt{g_{11}} u^1, \sqrt{g_{22}} u^2, 0, 0)$ are the coordinates of a point on the osculating plane to the curve $C: u^1 = u^1(s)$, at P_x , and from the third of equations (53) that another point on this osculating plane has coordinates

$$\left[\frac{1}{2} \sqrt{g_{11}} (\Gamma_{ij}^1 u^i u^j + u''^1), \frac{1}{2} \sqrt{g_{22}} (\Gamma_{ij}^2 u^i u^j + u''^2), \frac{1}{2} b_{(3)ij} u^i u^j, \frac{1}{2} b_{(4)ij} u^i u^j \right].$$

Then, if ρ is a parameter, a line of points on this osculating plane is given by

$$X_m = \frac{1}{2} \sqrt{g_{mm}} (\Gamma_{ij}^m u^i u^j + u''^m + \rho u'^m), \quad (76)$$

$$X_\sigma = \frac{1}{2} b_{(\sigma)ij} u^i u^j.$$

Elimination of the parameter ρ from equations (76) yields three equations of a line for every direction on the surface V_s at P_x , namely,

$$\delta_{12}^{mn} \left(2 \frac{X_m}{\sqrt{g_{mm}}} u^n + u''^m c^n \right) = 0, \quad (77)$$

$$2 X_\sigma = b_{(\sigma)ij} u^i u^j \quad (\sigma = 3, 4),$$

in which $c^n \equiv \Gamma_{ij}^n u^i u^j + u''^n$ ($n = 1, 2$). If the curve C is a geodesic on the surface V_s , the quantities denoted by c^n are identically zero and the coordinates of any point on the line (76) are given by

$$(78) \quad \frac{X_l}{\sqrt{g_{ll}}} = \rho \frac{u^l}{2}, \quad 2 X_\tau = b_{(\tau)ij} u^i u^j.$$

On eliminating ρ , u^{i1} , and u^{i2} from equations (78), we find

$$(79) \quad \sum_{\sigma \tau}^{34} X_{\sigma} b_{(\tau)ij} \frac{X_i X_j}{\sqrt{g_{ik} g_{jl}}},$$

which is the locus of the osculating planes to the geodesics on the surface V_2 at P_X , under the supposition that the surface sustains an orthogonal conjugate net. We replace the X 's in (79) by ξ 's to preserve the local coordinate notation and write (79) in notation (A) to obtain

$$(80) \quad \xi_3 \left(\frac{\xi_1^2}{R_4} + \frac{\xi_2^2}{R_4'} \right) - \xi_4 \left(\frac{\xi_1^2}{R_3} + \frac{\xi_2^2}{R_3'} \right) = 0.$$

This is the form which the cubic cone (74) takes with $D_3' = D_4' = 0$. The tangent plane to the surface V_2 at P_X lies on this cone and the normal plane to the surface V_2 at P_X is a double generator of the cone.

We now determine the locus of the normals through P_X to the tangent hyperplanes to the cone (80). The current coordinates X_{α} of a point on the normal through P_X to the tangent hyperplane to the cubic cone (80) at the point with coordinates ξ_{α} , are proportional to the partial derivatives of (80) with respect to ξ_{α} . These partial derivatives may be written, with the proportionality factor taken as unity, in the form

$$\begin{aligned} X_1 &= 2\xi_1 \left(\frac{\xi_1}{R_4} - \frac{\xi_4}{R_3} \right), & X_2 &= 2\xi_2 \left(\frac{\xi_2}{R_4'} - \frac{\xi_4}{R_3'} \right), \\ X_3 &= \frac{\xi_1^2}{R_4} + \frac{\xi_2^2}{R_4'}, & X_4 &= -\frac{\xi_1^2}{R_3} - \frac{\xi_2^2}{R_3'}. \end{aligned}$$

From these are obtained

$$R_4 X_3 + R_3 X_4 = \xi_2^2 \left(\frac{R_4}{R_4'} - \frac{R_3}{R_3'} \right), \quad R_4' X_3 + R_3' X_4 = \xi_1^2 \left(\frac{R_4'}{R_4} - \frac{R_3'}{R_3} \right),$$

$$\frac{R_3 X_1}{\xi_1} - \frac{R_2' X_2}{\xi_2} = 2 \xi_3 \left(\frac{R_3}{R_4} - \frac{R_3'}{R_4'} \right), \quad \frac{R_4 X_1}{\xi_1} - \frac{R_4' X_2}{\xi_2} = 2 \xi_4 \left(\frac{R_4'}{R_3} - \frac{R_4}{R_3} \right).$$

Using these four relations, together with (80), since ξ_α are the coordinates of a point on the cone (80), we find on eliminating ξ_α ,

$$X_3 \left(\frac{X_2^2}{R_3} + \frac{X_1^2}{R_3'} \right) + X_4 \left(\frac{X_2^2}{R_4} + \frac{X_1^2}{R_4'} \right) = 0,$$

and on replacing the X 's by ξ 's, this gives

$$(81) \quad \xi_3 \left(\frac{\xi_1^2}{R_3'} + \frac{\xi_2^2}{R_3} \right) + \xi_4 \left(\frac{\xi_1^2}{R_4'} + \frac{\xi_2^2}{R_4} \right) = 0.$$

The cubic cone (75) reduces to the form (81) in case $D_3' = D_4' = 0$. We notice that the cubic cones (80) and (81) coincide at a point of the surface for which

$$\frac{R_4}{R_3'} = \frac{R_4'}{R_3} = \frac{R_3}{-R_4'} = \frac{R_3'}{-R_4}.$$

As a consequence of these relations, we have

$$(82) \quad R_3^2 + R_4^2 = 0, \quad R_3'^2 + R_4'^2 = 0, \quad R_3 R_3' + R_4 R_4' = 0.$$

These conditions will be used in Section VIII.

VIII. OSCULATING REGULI. A CERTAIN CONGRUENCE

We consider next the osculating regulus determined by three successive v-tangents to the normal projection surface in the space $\xi_\sigma = 0$ ($\sigma = 3, 4$), at points of a fixed u-curve. The coordinates X^α of a point on the u-curve near P_x on the surface V_2 are given by the expansions

$$(83) \quad \bar{X}^\alpha = X^\alpha + X^\alpha_u \Delta u + \frac{1}{2} X^\alpha_{uu} \Delta u^2 + \frac{1}{6} X^\alpha_{uuu} \Delta u^3 + \dots,$$

and the v-tangent at the point with coordinates X^α is determined by this point and the point with coordinates X^α_v , where

$$(84) \quad \bar{X}^\alpha_v = X^\alpha_v + X^\alpha_{uv} \Delta u + \frac{1}{2} X^\alpha_{uuv} \Delta u^2 + \frac{1}{6} X^\alpha_{uuuv} \Delta u^3 + \dots.$$

If Y^α are the coordinates of a variable point on this v-tangent at X^α , and if t is a scalar parameter, then

$$\begin{aligned} Y^\alpha &= \bar{X}^\alpha + t \bar{X}^\alpha_v = X^\alpha + t X^\alpha_v + (X^\alpha_u + t X^\alpha_{uv}) \Delta u + \\ &\quad \frac{1}{2} (X^\alpha_{uu} + t X^\alpha_{uuv}) \Delta u^2 + \frac{1}{6} (X^\alpha_{uuu} + t X^\alpha_{uuuv}) \Delta u^3 + \dots \\ (85) \quad &= X^\alpha + t X^\alpha_v + X^\alpha_u \Delta u + t (\Gamma_{12}^h X^\alpha_h + b_{(\sigma)12} \eta^\alpha_{(\sigma)}) \Delta u + \\ &\quad \frac{1}{2} (\Gamma_{11}^h X^\alpha_h + b_{(\sigma)11} \eta^\alpha_{(\sigma)}) \Delta u^2 + \frac{t}{6} P (I_{112}^\sigma \eta^\alpha_{(\sigma)} + J_{112}^m X^\alpha_{,m}) \Delta u^2 + \dots \\ &= X^\alpha + \xi^m \frac{X^\alpha_{,m}}{\sqrt{g_{mm}}} + \xi^\sigma \eta^\alpha_{(\sigma)} \equiv X^\alpha + \xi^m \eta^\alpha_{(m)} + \xi^\sigma \eta^\alpha_{(\sigma)}, \end{aligned}$$

where the symbol P before an expression indicates the sum of terms obtained by permuting the subscripts cyclically.

From (85) we obtain the local coordinates of a point Y on the osculating regulus, namely,

$$\begin{aligned} \xi_1 &= \sqrt{q_{11}} \left[(1+t\Gamma_{12}^1) \Delta u + \frac{1}{2} (\Gamma_{11}^1 + \frac{t}{3} P J_{112}^1) \Delta u^2 + \dots \right], \\ (86) \quad \xi_2 &= \sqrt{q_{22}} \left[t + t\Gamma_{12}^2 \Delta u + \frac{1}{2} (\Gamma_{11}^2 + \frac{t}{3} P J_{112}^2) \Delta u^2 + \dots \right], \\ \xi_\sigma &= t b_{(\sigma)11} \Delta u + \frac{1}{2} (b_{(\sigma)11} + \frac{t}{3} P I_{112}^\sigma) \Delta u^2 + \dots \quad (\sigma = 3, 4), \end{aligned}$$

where the indices on the ξ 's are lowered for convenience. If, in (86), we consider ξ_1 , ξ_2 , ξ_3 and neglect ξ_4 then we are considering the osculating regulus of the orthogonal projection of the net on V_2 , onto the hyperplane with equation $\xi_4 = 0$. A similar remark can be made with 3 and 4 interchanged.

The local equation of the general quadric through P_X may be written as

$$(87) \quad a\xi_1^2 + b\xi_1\xi_2 + c\xi_2^2 + d\xi_1\xi_\sigma + e\xi_2\xi_\sigma + f\xi_\sigma^2 + g\xi_1 + h\xi_2 + \xi_\sigma = 0.$$

On requiring that the series (86) satisfy (87) identically in t and Δu to terms of the second order in Δu , it is found that the coefficients of (87) are given by

$$\begin{aligned} 2ag_{11} &= -b_{(\sigma)11}, \quad b\sqrt{q_{11}q_{22}} = -b_{(\sigma)12}, \quad c=h=g=0, \\ (88) \quad d\sqrt{q_{11}} b_{(\sigma)12} &= \frac{1}{2} b_{(\sigma)11} \Gamma_{12}^1 + \frac{1}{2} b_{(\sigma)12} \Gamma_{11}^1 + b_{(\sigma)12} \Gamma_{12}^2 - \frac{1}{6} P I_{112}^\sigma, \\ e\sqrt{q_{22}} &= \Gamma_{12}^1, \quad 2f b_{(\sigma)12} = \frac{1}{3} P J_{112}^1 - \Gamma_{11}^1 \Gamma_{12}^1 - 2\Gamma_{12}^1 \Gamma_{12}^2, \end{aligned}$$

so that the quadric (87) has the form

$$\begin{aligned}
 & 2\sqrt{g_{11}} b_{(\sigma)12}^2 \xi_1 \xi_2 + \sqrt{g_{22}} b_{(\sigma)11} b_{(\sigma)12} \xi_1^2 - g_{11} \sqrt{g_{22}} \left[\frac{1}{3} P J_{112}^1 - \Gamma_{11}^1 \Gamma_{12}^1 \right. \\
 (89) \quad & - 2 \Gamma_{12}^1 \Gamma_{12}^2 \left. \right] \xi_1^2 - \sqrt{g_{11} g_{22}} \left[b_{(\sigma)11} \Gamma_{12}^1 + b_{(\sigma)12} (\Gamma_{11}^1 + 2 \Gamma_{12}^2) - \frac{1}{3} P I_{112}^\sigma \right] \xi_1 \xi_\sigma \\
 & - 2 g_{11} b_{(\sigma)12} \Gamma_{12}^1 \xi_2 \xi_\sigma - 2 g_{11} \sqrt{g_{22}} b_{(\sigma)12} \xi_\sigma = 0.
 \end{aligned}$$

In case the surface V_2 sustains an orthogonal conjugate net, we have, besides $g_{12} = g_{21} = 0$, $b_{(\sigma)12} = b_{(\sigma)21} = 0$ ($\sigma = 3, 4$). Then the regulus (89) takes the form

$$(90) \quad (b_{(\sigma)11} \Gamma_{12}^1 - \frac{1}{3} P I_{112}^\sigma) \xi_1 \xi_\sigma + \sqrt{g_{11}} \left(\frac{1}{3} P J_{112}^1 - \Gamma_{11}^1 \Gamma_{12}^1 - 2 \Gamma_{12}^1 \Gamma_{12}^2 \right) \xi_\sigma^2 = 0.$$

In this case, the ruled surface, constituted by the tangents to the v -curves at points of a fixed u -curve, is developable, and the lines of the osculating regulus lie in two planes, namely,

$$\begin{aligned}
 & \xi_\sigma = 0, \\
 (91) \quad & (b_{(\sigma)11} \Gamma_{12}^1 - \frac{1}{3} P I_{112}^\sigma) \xi_1 + \sqrt{g_{11}} \left(\frac{1}{3} P J_{112}^1 - \Gamma_{11}^1 \Gamma_{12}^1 - 2 \Gamma_{12}^1 \Gamma_{12}^2 \right) \xi_\sigma = 0,
 \end{aligned}$$

with $\xi_\tau = 0$, $\tau \neq \sigma$.

In notation (A) the equations of these planes have the form

$$\begin{aligned}
 & \xi_\sigma = 0, \\
 (92) \quad & \sqrt{E} \left(\frac{D_\sigma}{E} + \frac{D_\sigma''}{G} \right) \xi_1 + 3 \left(\log \frac{E_v}{E \sqrt{G}} \right)_u \xi_\sigma = 0,
 \end{aligned}$$

with $\xi_\tau = 0$, $\tau \neq \sigma$.

By a suitable change of notation the osculating regulus determined by three successive u-tangents to V_2 at points of a fixed v-curve can be written by use of (89). Then again, in case $b_{(\sigma)12} = 0$, the planes corresponding to (92) have the equations

$$(93) \quad \xi_\sigma = 0, \\ \sqrt{G} \left(\frac{D_\sigma''}{G} + \frac{D_\sigma}{E} \right) \xi_2 + 3 \left(\log \frac{G_u}{G\sqrt{E}} \right)_v \xi_\sigma = 0,$$

with $\xi_\tau = 0$, $\tau \neq \sigma$. The second configuration of (92) and the second of (93) intersect in the line with direction numbers given by

$$(94) \quad \xi_1 : \xi_2 : \xi_\sigma = 3\sqrt{G} \left(\log \frac{E_v}{E\sqrt{G}} \right)_u : 3\sqrt{E} \left(\log \frac{G_u}{G\sqrt{E}} \right)_v : -\sqrt{EG} \left(\frac{1}{R_\sigma} + \frac{1}{R'_\sigma} \right),$$

R_σ and R'_σ being defined in (21). For $\sigma = 4$, this direction is in the space $\xi_3 = 0$. Since¹

$$\frac{1}{f_{gu}} = -\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v} = -\frac{1}{\sqrt{EG}} \frac{E_v}{2\sqrt{E}} = -\frac{E_v}{2E\sqrt{G}},$$

then

$$\frac{E_v}{E\sqrt{G}} = -\frac{2}{f_1}, \quad \frac{G_u}{G\sqrt{E}} = -\frac{2}{f_2},$$

where f_1 and f_2 are the radii of geodesic curvature of the parametric curves on the surface V_2 at P_X . Thus, (94) may be written

$$\xi_1 : \xi_2 : \xi_\sigma = 3\sqrt{G} \left[\log \frac{-2}{f_1} \right]_u : 3\sqrt{E} \left[\log \frac{-2}{f_2} \right]_v : -\sqrt{EG} \frac{R_\sigma + R'_\sigma}{R_\sigma R'_\sigma},$$

¹L. P. Eisenhart, Differential Geometry, Ginn and Co., 1909, p. 134.

or

$$(95) \quad \xi_1 : \xi_2 : \xi_\sigma = \frac{3\rho_{1u}}{\sqrt{E}\rho_1} : \frac{3\rho_{2v}}{\sqrt{G}\rho_2} : M_\sigma.$$

Thus, in the space $\xi_4 = 0$, we have the direction

$$(96) \quad \left(\frac{3\rho_{1u}}{\sqrt{E}\rho_1} : \frac{3\rho_{2v}}{\sqrt{G}\rho_2} : M_3 : 0 \right)$$

and in the space $\xi_3 = 0$, the direction

$$(97) \quad \left(\frac{3\rho_{1u}}{\sqrt{E}\rho_1} : \frac{3\rho_{2v}}{\sqrt{G}\rho_2} : 0 : M_4 \right).$$

The plane determined by the lines (96) and (97) through P_x , intersects the normal plane to V_2 at P_x in a line passing through the point with coordinates

$$(98) \quad (0, 0, M_3, -M_4).$$

One such normal line is obtained at every point of the surface, so there are ∞^2 of these normals. It can be shown that these ∞^2 normals constitute a congruence of normals to the surface V_2 , that is, that these lines can be assembled into a single infinity of developable surfaces in two ways. The development of this is omitted here. Let us denote the congruence generated by the line joining the point P_x to the point with coordinates (98) by Γ . It may be noted here that the unique normal to the surface V_2 at P_x , perpendicular to the normal determined by (98), that is, the line with direction $(0 : 0 : M_4 : M_3)$, generates a congruence normal

to the congruence Γ .

We shall now investigate the condition for the congruence Γ to be conjugate to the parametric net on V_2 , that is, the condition that the developables of the congruence Γ intersect V_2 in the parametric orthogonal conjugate net. The coordinates y^α of any point P_y on the line determined by P_x and the point with coordinates (98) may be expressed by

$$(99) \quad y^\alpha = x^\alpha + r d_\sigma \eta_{(\sigma)}^\alpha,$$

where $d_3 = M_3$, $d_4 = -M_4$, and r is a parameter. Now, if the congruence Γ is conjugate to the surface V_2 , P_y generates a surface S_y with the property that the tangents to the curves on S_y corresponding to the curves of the parametric net on V_2 , and the tangents to the curves of this net on V_2 , are coplanar. Differentiating (99) we obtain

$$(100) \quad y_{,i}^\alpha = x_{,i}^\alpha + r_{,i} d_\sigma \eta_{(\sigma)}^\alpha + r d_{\sigma,i} \eta_{(\sigma)}^\alpha + r d_\sigma \eta_{(\sigma),i}^\alpha.$$

In consequence of the second of the differential equations (4), this can be written

$$(101) \quad \begin{aligned} y_{,i}^\alpha &= x_{,i}^\alpha + r_{,i} d_\sigma \eta_{(\sigma)}^\alpha + r d_{\sigma,i} \eta_{(\sigma)}^\alpha - r d_\sigma b_{(\sigma)li} g^{lm} x_{,m}^\alpha + \\ & r d_\sigma \nu_{(\tau\sigma)i} \eta_{(\tau)}^\alpha = x_{,i}^\alpha (1 - r d_\sigma b_{(\sigma)lm} g^{li}) + \\ & \frac{r_{,i}}{r} (y^\alpha - x^\alpha) + r \eta_{(\sigma)}^\alpha (d_{\sigma,i} + d_\tau \nu_{(\sigma\tau)i}). \end{aligned}$$

Since, for $i = 1$ or $i = 2$, the points with coordinates $x_{,i}^\alpha$, $y_{,i}^\alpha$, x^α , y^α must be coplanar, the coefficient of $\eta_{(\sigma)}^\alpha$ must vanish. That is,

$$(102) \quad d_{\sigma,i} + d_\tau \nu_{(\sigma\tau)i} = 0.$$

Also, we may write

$$(103) \quad d_{\tau,i} + d_{\sigma} \nu_{(\tau\sigma)i} = 0.$$

On multiplying (102) by d_{σ} and (103) by d_{τ} and then adding, we find, in consequence of (5),

$$(104) \quad d_{\sigma} d_{\sigma,i} + d_{\tau} d_{\tau,i} = 0 \quad (\sigma=3,4) \quad (\sigma \neq \tau).$$

It follows that

$$(105) \quad d_{\sigma}^2 + d_{\tau}^2 = \text{const.} \quad (\sigma \neq \tau).$$

Therefore, a necessary and sufficient condition that the line determined by the point P_x and the point with coordinates (98) generate a congruence conjugate to the surface V_s , is that

$$(106) \quad M_3^2 + M_4^2 = \text{const.}$$

We observe that the congruence orthogonal to Γ is conjugate to V_s in case Γ is conjugate to V_s .

The results of this section thus far, may be gathered together in the following theorem. For a surface which sustains an orthogonal conjugate net, the lines of the osculating regulus, in the space $\xi_4 = 0$, determined by three successive tangents to the v-curves at points of a fixed u-curve through P_x , lie in two planes, namely, the tangent plane to the surface at P_x and a residual plane; the lines of the osculating regulus, in the space $\xi_4 = 0$, determined by three successive tangents to the u-curves at points of a fixed v-curve through P_x , lie in two planes, namely, the tangent plane to the surface at P_x and a residual plane. These two residual planes have a line of intersection in the space $\xi_4 = 0$. There is an analogous line in the space $\xi_3 = 0$. The

plane of these two lines intersects the normal plane to the surface at P_x in a line which generates a congruence Γ as P_x varies over the surface. The congruence Γ is conjugate to the surface if, and only if, the sum of the squares of the mean curvatures of the normal projection surfaces $V_{2(\sigma)}$ is constant.

At this point we recall the conditions (82) which follow from the conditions under which the reciprocal cubic cones of Section VII coincide at a point of a surface which sustains an orthogonal conjugate net. If (82) hold, then the condition (106) is satisfied with the constant equal to zero. Therefore, we may state that a surface in space S_4 which sustains an orthogonal conjugate net and which has the property that the reciprocal cubic cones coincide at every point of it, is such that the congruence Γ is conjugate to it.

To close this section we find the equation of the net on the surface V_2 for which the congruence Γ is the intersector normal congruence. Let us drop the restriction that the surface sustain a conjugate net, and replace the ratio ξ_3 / ξ_4 in the third of equations (61) by the ratio $M_3 / -M_4$ to obtain the net

$$(107) \quad (M_3 D_4 + M_4 D_3) du^2 + 2(M_3 D_4' + M_4 D_3') du dv + (M_3 D_4'' + M_4 D_3'') dv^2 = 0.$$

The harmonic invariant of the quadratic forms in (72) and (107) is zero. Therefore, we may state that the directions of the curves of the net (107) separate the directions of the limiting intersector net harmonically.

The net (107) is indeterminate in case the normal projection surfaces $V_{2(\sigma)}$ are minimal.

If the surface V_2 is immersed in S_3 , that is, if the matrix $(D_\sigma, D_\sigma', D_\sigma'')$ is of rank one, then the net (107) becomes

$$(108) \quad (M_3 D_4 + M_4 D_3) (D_3 du^2 + 2 D_3' du dv + D_3'' dv^2) = 0,$$

which gives the asymptotics on the surface, since the first factor of the left member of this equation does not vanish identically.

IX. THE OSCULATING HYPERPLANE TO A CURVE ON THE SURFACE

Attention is now directed to the osculating hyperplane to the curve C on the surface V_s at the point P_x . From the equation of this hyperplane we can deduce the differential equation of the quasi-asymptotic geodesics on the surface at P_x .

Let us write the general equation of a hyperplane through P_x in local coordinates in the form

$$(109) \quad A_\ell \xi^\ell + A_\sigma \xi^\sigma = 0$$

and require that the series (50), in Section VI, satisfy (109) identically to terms of the third order in Δs . The conditions to be satisfied are

$$(110) \quad \begin{aligned} A_\ell \sqrt{g_{\ell\ell}} u'^\ell &= 0, \\ A_\ell \sqrt{g_{\ell\ell}} (\Gamma_{ij}^\ell u'^i u'^j + u''^\ell) + A_\sigma b_{(\sigma)ij} u'^i u'^j &= 0, \\ A_\ell \sqrt{g_{\ell\ell}} (J_{ijk}^\ell u'^i u'^j u'^k + 3\Gamma_{ij}^\ell u''^i u'^j + u'''^\ell) + A_\sigma (I_{ijk}^\sigma u'^i u'^j u'^k + 3b_{(\sigma)ij} u''^i u'^j) &= 0. \end{aligned}$$

Elimination of A_ℓ and A_σ from (109) and (110) gives the equation of the osculating hyperplane to C at P_x , namely,

$$(111) \quad \left| \begin{array}{cc} \xi^\ell & \xi^\sigma \\ \sqrt{g_{\ell\ell}} u'^\ell & 0 \\ \sqrt{g_{\ell\ell}} (\Gamma_{ij}^\ell u'^i u'^j + u''^\ell) & b_{(\sigma)ij} u'^i u'^j \\ \sqrt{g_{\ell\ell}} (J_{ijk}^\ell u'^i u'^j u'^k + 3\Gamma_{ij}^\ell u''^i u'^j + u'''^\ell) & I_{ijk}^\sigma u'^i u'^j u'^k + 3b_{(\sigma)ij} u''^i u'^j \end{array} \right| = 0,$$

where $\ell = 1, 2$ for the first and second columns of the determinant and $\sigma = 3, 4$ for the third and fourth columns.

For a generic curve C on V_2 at P_X , the tangent plane to the surface at P_X and the osculating hyperplane to C at P_X determine a space of four dimensions. We ask for the curves on V_2 through P_X which have the property that the tangent plane to V_2 at P_X and the osculating hyperplane to any one of these curves at P_X , determine a space of three dimensions. Bompiani¹ calls such curves "quasi-asymptotic curves". If the osculating hyperplane given by (111) is to contain the tangent plane to V_2 at P_X , then the lower right-hand second order minor of the determinant (111) must vanish. This gives the differential equation of the quasi-asymptotic curves on the surface, namely, (after expanding),

$$(112) \quad \delta_{34}^{\sigma\tau} (b_{(\sigma)\ell m} \Gamma_{ij\kappa}^{\tau} u^i u^j u^{\kappa} u^{\ell} u^m + 3 b_{(\sigma)\ell m} b_{(\tau)ij} u^i u^j u^{\ell} u^m) = 0.$$

If the quasi-asymptotic curves are also geodesics, then

$$u''^h = -\Gamma_{ij}^h u^i u^j \quad \text{and (112) becomes}$$

$$(113) \quad \delta_{34}^{\sigma\tau} (b_{(\sigma)\ell m} \Gamma_{ij\kappa}^{\tau} - 3 \Gamma_{ij}^h b_{(\sigma)\ell m} b_{(\tau)h\kappa}) u^i u^j u^{\kappa} u^{\ell} u^m = 0.$$

Thus, through every point on the surface V_2 there pass five quasi-asymptotic geodesic curves, at least one of which must be real.

Let us write (113) in notation (A) with the supposition that the surface V_2 sustains an orthogonal conjugate net. The five quasi-asymptotic geodesic directions at P_X are then given by the vanishing of a quintic differential form, namely,

¹E. Bompiani, Contributo alla geometria proiettivo-differenziale di una superficie, Boll. d. U. M. It., III, 49 (1924), p. 313.

$$\begin{aligned}
& [D_4 D_{3u} - D_3 D_{4u} - A_3 (D_3^2 + D_4^2)] du^5 + \frac{3E_v}{2G} (D_4 D_3'' - D_3 D_4'') du^4 dv + \\
& [(D_3 D_4'' - D_4 D_3'') (\log \frac{G}{E})_u - A_3 (D_3 D_3'' + D_4 D_4'') + (D_4'' D_{3u} - D_3'' D_{4u})] du^3 dv^2 + \\
(114) \quad & [(D_3 D_4'' - D_4 D_3'') (\log \frac{G}{E})_v - B_3 (D_3 D_3'' + D_4 D_4'') + (D_4 D_{3v}'' - D_3 D_{4v}'')] du^2 dv^3 + \\
& \frac{3G_u}{2E} (D_4 D_3 - D_3 D_4) du dv^4 + [D_4'' D_{3v}'' - D_3'' D_{4v}'' - B_3 (D_3''^2 + D_4''^2)] dv^5 = 0.
\end{aligned}$$

If the parametric curves on the surface are quasi-asymptotic geodesics, then the coefficients of du^5 and dv^5 in (114) must vanish, so that

$$\begin{aligned}
A_3 &= \frac{D_4 D_{3u} - D_3 D_{4u}}{D_3^2 + D_4^2} = \frac{\partial}{\partial u} \arctan \frac{D_3}{D_4}, \\
(115) \quad B_3 &= \frac{D_4'' D_{3v}'' - D_3'' D_{4v}''}{D_3''^2 + D_4''^2} = \frac{\partial}{\partial v} \arctan \frac{D_3''}{D_4''}.
\end{aligned}$$

Because the surface sustains an orthogonal conjugate net we have $D_3' = D_4' = 0$, and in this case the integrability condition (8') reduces to $A_{3v} = B_{3u}$. Then, from (115)

$$\frac{\partial^2}{\partial u \partial v} \left(\arctan \frac{D_3}{D_4} - \arctan \frac{D_3''}{D_4''} \right) = 0,$$

which may be written in the form

$$(116) \quad \frac{\partial^2}{\partial u \partial v} \arctan \frac{D_3 D_4'' - D_4 D_3''}{D_3 D_3'' + D_4 D_4''} = 0.$$

Therefore, the curves of the orthogonal conjugate net are quasi-asymptotic geodesics if, and only if, the angle between the lines

of the degenerate central conic is the same for every point on the surface. With $D_3' = D_4' = 0$, equation (56) reduces to $D_3 D_3'' + D_4 D_4'' = 0$, so that by (116) the angle between the lines of the degenerate central conic is $\pi/2$. The angle between the intersector normals relative to the parametric curves on the surface is also $\pi/2$ in this case.

If the curves of the orthogonal conjugate net enjoy the property of being quasi-asymptotic geodesic curves on the surface V_2 , then (114) reduces, in consequence of (115), to

$$(D_3 D_4'' - D_4 D_3'')(GE_u du - EG_v dv) + EG[(D_3'' D_{4u} - D_4'' D_{3u})du + (D_3 D_{4v}'' - D_4 D_{3v}'')dv] = 0,$$

which may be expressed in the form

$$(117) \quad (\log EN)_u du - (\log GN)_v dv = 0,$$

where $N \equiv (D_3 D_4'' - D_4 D_3'')^{-1}$. Therefore, the following theorem can be announced. If a surface V_2 in space S_4 sustains an orthogonal conjugate net and if the curves of this net are quasi-asymptotic geodesics, then, besides these (counted twice), there is a unique quasi-asymptotic geodesic curve at each point of the surface with direction given by (117).

Let us study next the line of intersection of the osculating hyperplane (111) to any curve C on the surface V_2 with the normal plane to V_2 at P_x . If $\xi_1 = \xi_2 = 0$, and if the simplifying substitutions

$$(118) \quad \begin{aligned} L^l &\equiv J_{ij\kappa}^l u^i u^j u^\kappa + 3 \Gamma_{ij}^l u^i u^j + u^{m^l}, & C^l &\equiv \Gamma_{ij}^l u^i u^j + u^{m^l}, \\ T^\sigma &\equiv I_{ij\kappa}^\sigma u^i u^j u^\kappa + 3 b_{(\sigma)ij} u^i u^j, & \varphi_\sigma &\equiv b_{(\sigma)ij} u^i u^j, \end{aligned}$$

are used, then the left member of (111) becomes

$$(c^2 u^1 - c^4 u^2)(T^4 \xi_3 - T^3 \xi_4) - (L^2 u^1 - L^4 u^2)(\varphi_4 \xi_3 - \varphi_3 \xi_4).$$

Thus, the equations of the line of intersection of the normal plane to V_2 at P_X with the osculating hyperplane to the curve C at P_X , are

$$(119) \quad \xi_1 = 0, \quad \xi_2 = 0, \quad (\delta_{ij}^{12} u^i c^j)(\delta_{34}^{\sigma\tau} T^\sigma \xi_\tau) - (\delta_{ij}^{12} u^i L^j)(\delta_{34}^{\sigma\tau} \varphi_\sigma \xi_\tau) = 0.$$

If C is a geodesic on the surface V_2 , the quantities c^j vanish and the equations of the line (119) become

$$(120) \quad \xi_1 = 0, \quad \xi_2 = 0, \quad (\delta_{ij}^{12} u^i L^j)(\delta_{34}^{\sigma\tau} \varphi_\sigma \xi_\tau) = 0.$$

If the first factor in the third equation of (120) is not zero, the line of intersection of the normal plane to V_2 at P_X and the osculating hyperplane to the geodesic C , with the equations $u^i = u^i(s)$ at P_X , has the equations

$$(121) \quad \xi_1 = 0, \quad \xi_2 = 0, \quad \delta_{34}^{\sigma\tau} \xi_\sigma b_{(\tau)lm} u^l u^m = 0.$$

This is the same as the line (60). Hence, the line of intersection of the osculating hyperplane to a geodesic curve on V_2 at P_X and the normal plane to V_2 at P_X is the intersector normal corresponding to this geodesic.

If the factor $\delta_{ij}^{12} u^i L^j$ in (120) vanishes, the osculating hyperplane to the geodesic curve C at P_X contains the normal plane to V_2 at P_X . Thus, the geodesics through P_X for which the osculating hyperplane contains the normal plane to V_2 at P_X are given by

$$(122) \quad \delta_{\lambda s}^{12} u^{\lambda} \left(J_{ij\kappa}^s - \Gamma_{ij}^h \Gamma_{h\kappa}^s - \frac{\partial \Gamma_{ij}^s}{\partial u^{\kappa}} \right) u^i u^j u^{\kappa} = 0.$$

On replacing $J_{ij\kappa}^s$ by its expression from (15) this becomes

$$(123) \quad \delta_{lm}^{12} \left(b_{(\sigma)ij} b_{(\sigma)p\kappa} g^{pm} \right) u^i u^j u^{\kappa} u^l = 0.$$

This differential equation gives four directions at P_x , the directions of the geodesics through P_x , the osculating 3-spaces of which contain the normal plane to V_x at P_x . Let us find the form of (123) in case the surface sustains an orthogonal conjugate net. Using notation (A) with $D_3' = D_4' = 0$, we find that (123) becomes

$$(124) \quad [(ED_3'' - GD_3)(D_3 du^2 + D_3'' dv^2) + (ED_4'' - GD_4)(D_4 du^2 + D_4'' dv^2)] du dv = 0.$$

Therefore, we may state that, if the curves of the orthogonal conjugate net at P_x are geodesics, there are four geodesics through P_x on V_x , the osculating 3-spaces for which, contain the normal plane. If the parametric curves, that is the curves of the conjugate net, are not geodesics, then there are only two geodesics through P_x , the osculating 3-spaces for which, contain the normal plane. These two geodesic tangents separate the directions of the orthogonal conjugate net harmonically.

In case the curves of the parametric conjugate net are geodesics, we have $E_v = 0$ and $G_u = 0$, so that E is a function of u alone and G is a function of v alone. Also, in this case, we have from (56) that $D_3 D_3'' + D_4 D_4'' = 0$. Using the integrability conditions (7') with $D_3' = D_4' = 0$, we readily find that

$$\frac{1}{2} G_u (M_3 D_3'' + M_4 D_4'') - (D_3'' D_{3u}'' + D_4'' D_{4u}'') = D_3'' D_4'' (A_3 + A_4) = 0,$$

so that

$$(125) \quad (D_3''^2 + D_4''^2)_u = G_u (M_3 D_3'' + M_4 D_4'').$$

Thus, if G is a function of v alone, then $D_3''^2 + D_4''^2$ is also a function of v alone. Similarly, if E is a function of u alone, then $D_3''^2 + D_4''^2$ is a function of u alone. On using the relation $D_3 D_3'' + D_4 D_4'' = 0$, we may write (124) as

$$(126) \quad \left(\frac{D_3''^2 + D_4''^2}{E} du^2 - \frac{D_3''^2 + D_4''^2}{G} dv^2 \right) du dv = 0$$

and from the form of (126) we see that, in this case, the four geodesics for which the osculating hyperplane at P_X contains the normal plane to V_2 at P_X , can be found by quadratures.

If we start again with (124) and impose the condition that V_2 be immersed in space S_3 , that is, that besides $D_3' = D_4' = 0$, we have also $D_3 D_4'' - D_4 D_3'' = 0$, we arrive at the equation

$$(127) \quad (D_3 du^2 + D_4 dv^2) du dv = 0,$$

after rejecting a non-zero factor. In this case two of the geodesics are straight lines on the surface, since if a curve on the surface is a geodesic and also an asymptotic it must be a straight line.

From (111) we can find the differential equation of the curves on V_2 at P_X along which the osculating hyperplane (111) coincides with the space $\xi_\sigma = 0$ ($\sigma = 3$ or 4). For these curves it is necessary that

$$(128) \quad b_{(\sigma)lm} u'^l u''^m = 0, \quad I_{lmn}^{\sigma} u'^l u''^m u'''^n + 3 b_{(\sigma)lm} u'^l u''^m = 0.$$

By differentiating the first of these two relations one obtains

$$\frac{\partial b_{(\sigma)lm}}{\partial u^n} u'^l u''^m u'''^n + 2 b_{(\sigma)lm} u'^l u''^m = 0,$$

so that

$$b_{(\sigma)lm} u'^l u''^m = -\frac{1}{2} \frac{\partial b_{(\sigma)lm}}{\partial u^n} u'^l u''^m u'''^n$$

and thus the second of (128) becomes

$$(129) \quad \left(\Gamma_{lm}^h b_{(\sigma)hn} + b_{(\tau)lm} \gamma_{(\sigma\tau)n} - \frac{1}{2} \frac{\partial b_{(\sigma)lm}}{\partial u^n} \right) u'^l u''^m u'''^n = 0.$$

Thus, there are three curves on the surface through P_x at points of which the osculating hyperplane coincides with the 3-space with the equation $\xi_3 = 0$. Also, there are 3 curves through each point of V_2 at the points of which the osculating hyperplane coincides with the 3-space with the equation $\xi_4 = 0$.

If we adopt notation (A) and suppose that the surface sustains an orthogonal conjugate net, then (129) may be written in the form

$$(130) \quad D_4(\log R_4)_u du^3 + 3[D_4(\log R_4)_v du + D_4''(\log R_4')_u dv] du dv + D_4''(\log R_4')_v dv^3 + 2(A_3 D_3 du^3 + B_3 D_3'' dv^3) = 0.$$

Along the curves given by this equation the osculating hyperplane coincides with the 3-space $\xi_4 = 0$. By interchanging 3 and 4 we find from (130) the differential equation of the curves on V_2 along which the osculating hyperplane is the space $\xi_3 = 0$.

It may be observed that if R_4 and R_4' are constant, equation (130) reduces to

$$(131) \quad A_3 D_3 du^3 + B_3 D_3'' dv^3 = 0.$$

By interchanging 3 and 4 in (130) and supposing R_3 and R_3' to be constant, we obtain

$$(132) \quad A_4 D_4 du^3 + B_4 D_4'' dv^3 = 0,$$

but since $A_3 + A_4 = 0$ and $B_3 + B_4 = 0$ (by (5)), this can be written in the form

$$(133) \quad A_3 D_4 du^3 + B_3 D_4'' dv^3 = 0.$$

The equations (131) and (133) represent the same curves if, and only if, $D_3 D_4'' - D_4 D_3'' = 0$, that is, if, and only if, the surface is immersed in space S_3 .

VITA

Charles Eugene Springer was born October 25, 1903, at Storm Lake, Iowa. He attended the High School in Shelbyville, Missouri, for three years, and graduated from the High School at Cleveland, Oklahoma, in June, 1921, after spending one year there. From the autumn of 1921 to the summer of 1926 he was a student at the University of Oklahoma, where he received the degree of Bachelor of Arts in June, 1925, and the degree of Master of Arts in June, 1926. During the following year he was at the same institution as instructor in mathematics and director of the University Band.

From October, 1927, to June, 1929, he was in residence at the University of Oxford as a Rhodes Scholar from Oklahoma. At Oxford he received instruction in mathematics from his tutor, Mr. Newbould, and at the same time attended lectures delivered by Professors Dixon, Hardy, Love, Nicholson, Thompson, and Veblen.

In the summer of 1929 and the succeeding autumn, winter, and summer quarters, and also in the summer quarters of 1931, 1932, and 1934, he was a student at the University of Chicago, where he studied under Professors Bartky, Barnard, Bliss, Dickson, Graves, Lane, Laves, and Lunn, and under visiting Professors Blaschke and Bompiani. During the spring of 1930 he was instructor in mathematics at Iowa State College. Since September, 1930, he has been a member of the mathematics staff at the University of Oklahoma.

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